"Constant perturbation"

A constant perturbing potential applied after a particular time

$$V = \begin{cases} 0 & t \leq 0 \\ V & t > 0 \end{cases}$$

Initial condition $c_n^{(0)} = \delta_{ni}$ $\Big\{$ i some specific state

$$c_n^{(1)} = \frac{-i}{\hbar} V_{ni} \int_0^t e^{i\omega_{ni}t'} dt'$$

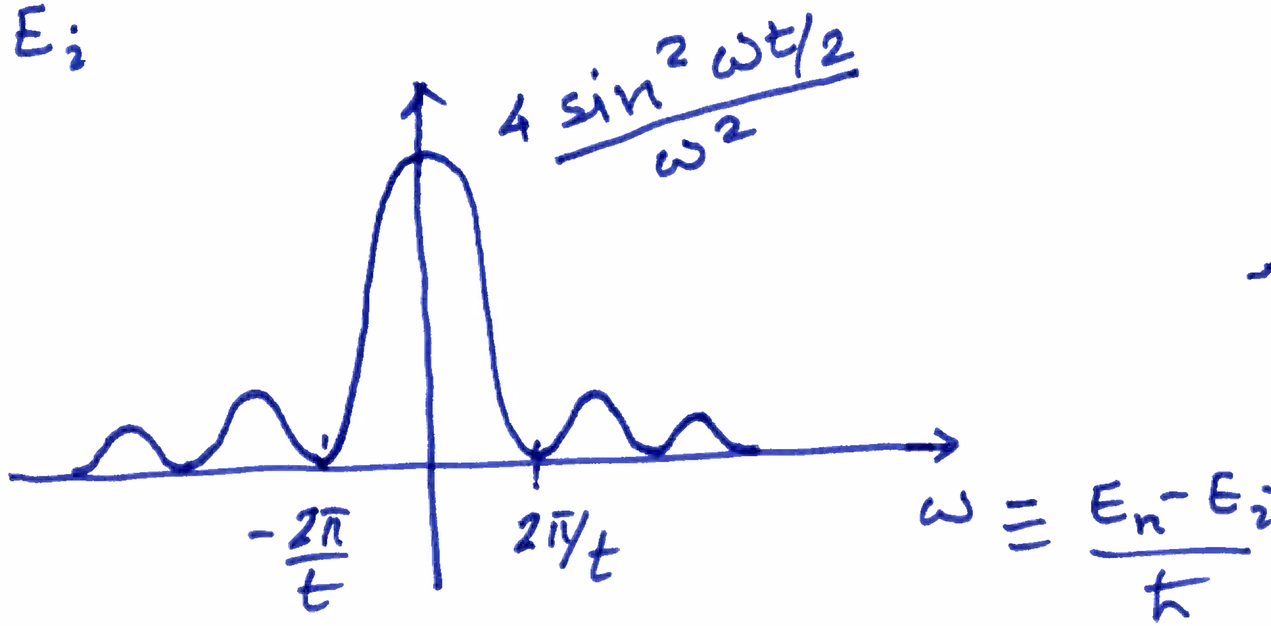
$$= -\frac{V_{ni}}{E_n - E_i}\left(e^{i\omega_{ni}t} - 1\right)$$

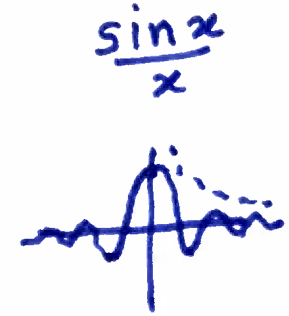
$e^{ix} - 1$

$= e^{ix/2}\left(e^{ix/2} - e^{-ix/2}\right)$

$$\therefore |c_n^{(1)}|^2 = \frac{4\,|V_{ni}|^2}{(E_n - E_i)^2} \sin^2\left[\frac{(E_n - E_i)t}{2\hbar}\right]$$

Plot $|c_n^{(1)}|^2$ at fixed t, as a function of $E_n - E_i$





Note ~~max~~ transition probability largest for $\omega \leq \frac{2\pi}{t}$ i.e. $\Delta E \Delta t \lesssim 2\pi\hbar$

[Note external V is a source of energy]

For large t, (large compared to intrinsic time scale $\frac{\hbar}{E_n - E_i}$ of the system)

The central peak becomes very sharp.

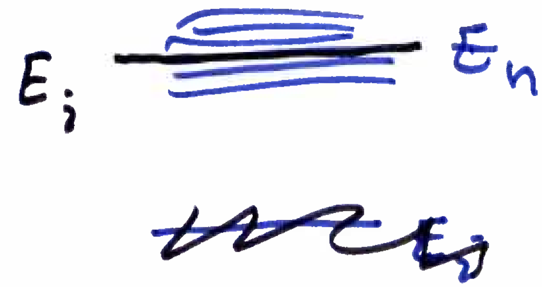
In fact
$$\frac{\sin^2 \omega t/2}{\omega^2} = \left(\frac{t}{2}\right)^2 \times \left(\frac{\sin \omega t/2}{\omega t/2}\right)^2$$

Then for levels with $\omega \to 0$ (E_n close to E_i)

the second factor $\to 1$ and $|C_n^{(0)}|^2 \sim t^2$

Interpretation requires some "physical mathematics"

We use the fact that there is a set of states (almost a continuum) into which the transitions must be occuring.



Assume level "density" $\rho(E)$

s.t. $\left.\begin{array}{l}\text{number of levels}\\ \text{in range } \Delta E\\ \text{at } E\end{array}\right\} = \rho(E)\Delta E$

Further, use the mathematical identity

$$\lim_{\alpha \to \infty} \frac{\sin^2 \alpha x}{\alpha x^2} = \pi \delta(x)$$

Thus, we restate the results as

$$\text{Total probability} = \int dE_n \rho(E_n) |C_n^{(1)}|^2$$

$$= 4 \int \sin^2\left\{\frac{(E_n - E_i)t}{2\hbar}\right\} \frac{|V_{ni}|^2}{(E_n - E_i)^2} \rho(E_n) dE_n$$

$$\lim_{t \to \infty} \to = \frac{4t}{2\hbar} \int \pi \delta(E_n - E_i) |V_{ni}|^2 \rho(E_n) dE_n$$

Thus the physically measurable quantity is proposed to be the rate of transitions,

$$W_{i\to\{n\}} \equiv \frac{d}{dt}\sum_n |C_n^{(0)}|^2 = \frac{2\pi}{\hbar}\,\overline{|V_{ni}|^2\rho(E_n)}\Big|_{n\simeq i}$$

The overbar is to indicate averaging over various other quantum numbers. Or we also write this as

$$W_{i\to\{n\}} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$$

"Fermi's Golden rule" (no. 2)

… when ρ is sharply peaked

Comment on $C^{(2)}$ for "constant" pert.

$$C_n^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \sum_m V_{nm} V_{mi} \int_0^t dt' e^{i\omega_{nm}t'} \int_0^{t'} e^{i\omega_{mi}t''} dt'' dt'$$

$$= \frac{i}{\hbar} \sum_m \frac{V_{nm} V_{mi}}{E_m - E_i} \int_0^t \left(e^{i\omega_{ni}t'} - e^{i\omega_{nm}t'}\right) dt'$$

.... if we throw away oscillatory time dependence we get

$$W_{i \to \{n\}} = \frac{2\pi}{\hbar} \left| V_{ni} + \sum_m \frac{V_{nm} V_{mi}}{E_i - E_m} \right|^2 \rho(E_n) \Big|_{E_n \simeq E_i}$$