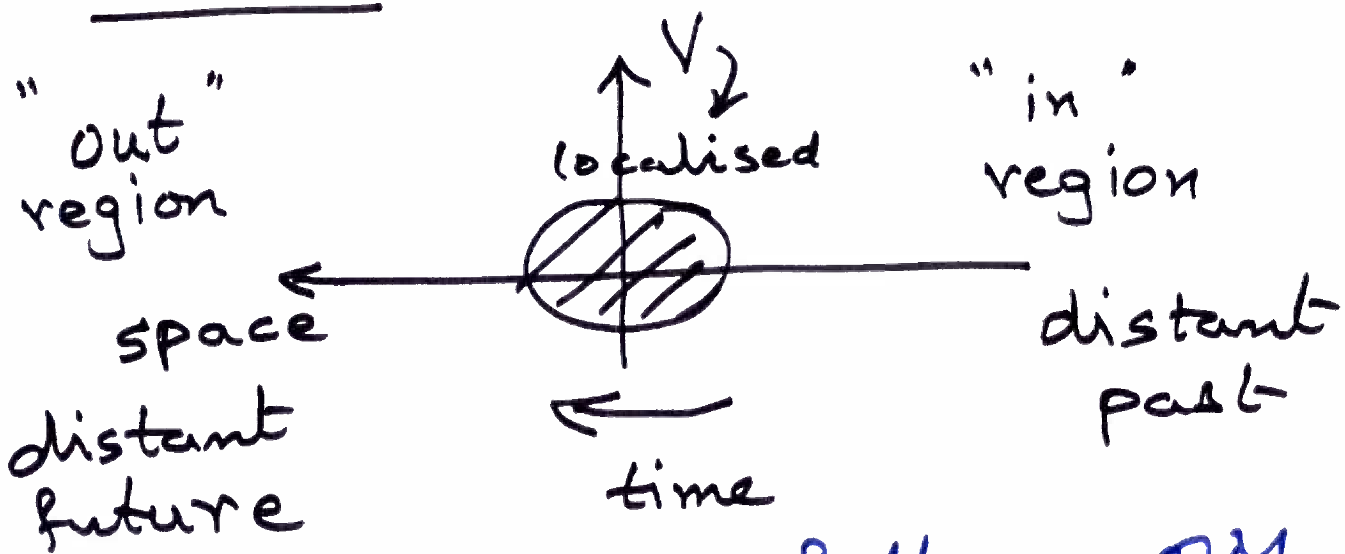
S-matrix and phase shifts:

↳ Scattering matrix

Overview:



$$H = H_0 + V$$

The picture follows QM convention

$$\langle \beta^{out} | \ldots\ldots\ldots | \alpha^{in} \rangle$$

"out" basis "in" basis

Modify & specialise the time dependent perturbation theory

Important formulae (a trailer)

1. $$\langle \beta | S | \alpha \rangle = \langle \beta | \alpha \rangle - \frac{i}{\hbar} \int \phi_\beta^*(\vec{x}'t') V(\vec{x}'t') \psi_\alpha^+(\vec{x}'t') \, d^3x' \, dt'$$

Where $\vec{x}'$ is supposed to be in the "out" region though formalism is more general.

ψ_α^+ denotes as free "in" α state already propagated correctly upto $(\vec{x}', t')$

2. Use of Green Function or the kernel ... to define ψ^+_α

$$\psi(x_f t_f) = i\int G_0^+(x_f t_f; x_i t_i)\,\psi^s(x_i\, t_i)\, dx_i$$

$$+ \frac{1}{\hbar}\int dt'\, d^3x'\, G_0^+(x_f t_f\, x' t')\, V(x't')\, \psi^s(x't')$$

Use as ψ^+

where G_0^+ is the kernel for H_0 evolution,

"+" means propagating forward in time

Specialise & simplify

3. Stationary collision theory

$$(H_0 - E)\, u_\alpha(\vec{x}) = 0 \; ; \quad (H - E_\alpha)\, \chi_\alpha(\vec{x}) = 0$$

$$\phi_\alpha(\vec{x},t) = e^{-i\omega_\alpha t} u_\alpha(\vec{x})$$

$$\psi^+(\vec{x},t) = e^{-i\omega_\alpha t} \chi^+_\alpha(\vec{x})$$

$\Big\{$ Interaction picture ansatz to build in time dependence due to H_0

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4 Fermi's Golden Rule for collision theory:

$$\langle\beta|S - \mathbb{1}|\alpha\rangle = -\frac{i}{\hbar}\langle\beta|\underset{\uparrow}{T}|\alpha\rangle \int_{-\infty}^{\infty} \underset{\sim\sim}{g(t)}\, e^{i\omega_{\beta\alpha}t}\, dt$$

T-matrix

↳ not - so-important time dep.

$$\langle\beta|T|\alpha\rangle = \int u^*_\beta(\vec{x}')\, V(\vec{x}')\, \chi^+_\alpha(\vec{x}')\, d^3x'$$

$$W_{\beta\leftarrow\alpha} = \frac{2\pi}{\hbar}\rho(\beta)\, |\langle\beta|T|\alpha\rangle|^2$$

S-matrix : (S-operator)

This is defined as the operator which maps eigenstates of H_0 in the remote past to those in remote future

$$|\beta^{out}\rangle = \text{"}\sum_{\alpha}\text{"} S_{\beta\alpha} |\alpha^{in}\rangle$$

where α, β are generic eigenstates of H_0.

- As a caution it is assumed that the basis $\{|\beta^{out}\rangle\}$ is isomorphic to, but not identical to, the basis $\{|\alpha^{in}\rangle\}$. // The two may differ by an overall phase

The S-matrix is unitary

$$S^{\dagger}_{\beta\gamma} S_{\gamma\alpha} = \delta_{\beta\alpha}$$

$$S_{\rho\sigma} S^{\dagger}_{\sigma\lambda} = \delta_{\rho\lambda}$$

↳ belong to same Hilbert space; the "~~out~~ in"

If the rate W or cross-section σ calculated from $T = S - \mathbb{1}$ do not seem to obey unitarity, then there is some aspect not accounted for, the Hilbert spaces $|out\rangle$ & $|in\rangle$ are not ~~iso~~ isomorphic $\Rightarrow$ "error" or "new physics"

Evolution equation for the Green Function

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Recall,

$$i\hbar \frac{\partial}{\partial t}|\alpha t\rangle_I = V_I(t)|\alpha t\rangle_I$$

We propose to use $|\alpha t\rangle_I$ as the ψ^+ or χ^+ states, and take them to be evolved using Green Function

$$\psi_\alpha(\vec{x}_f t_f) = i\int G^+(\vec{x}_f t_f \vec{x}_i t_i)\,\psi_\alpha(\vec{x}_i t_i)\,d^3x_i$$

Thus, the G^+ obeys the equation

$$i\hbar \frac{\partial}{\partial t} G^+ = V_I(t)\,G^+$$

... need to clarify "+"