Pre-view ....

Expect

$$\chi_\alpha^+ = u_\alpha(\vec{r}) + \underbrace{\frac{c}{r} f(\vec{k}_\alpha, \hat{r}) e^{i k_\alpha r}}$$

label of state

incoming state

$$\underbrace{\frac{e^{ikr}}{r}}_{\text{spherically symmetric part emerging at } \infty} \times f$$

↳ angular dependence & other detail

Note $\underbrace{\left| \frac{e^{ikr}}{r} \right|^2}_{\text{flux which gives a}} \times 4\pi r^2 \sim 4\pi$

probability density indep. of r → same for all spherical shells as $r \to \infty$

Returning to G.F. ... a mathematical derivation

$$\left(E_n^{(0)} + \frac{\hbar^2}{2m}\nabla^2\right) G(\vec{x}-\vec{x}') = \delta^3(\vec{x}-\vec{x}')$$

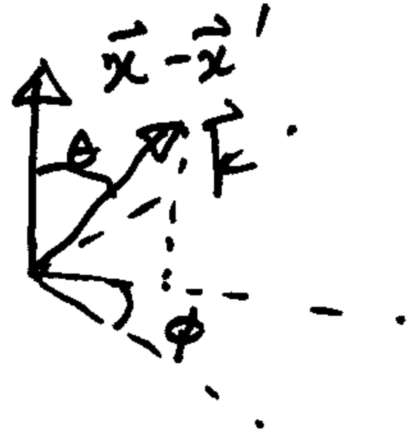
Then

$$G(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{E_n - \frac{\hbar^2|\vec{k}|^2}{2m} + i\epsilon}$$

... with ϵ taken to $\underline{0^+}$ eventually
(i.e. $\epsilon > 0$)

Let $d^3k = k^2 dk \sin\theta \, d\theta \, d\phi$

Where θ, ϕ are angles
w.r.t. the fixed vector $\vec{x}-\vec{x}'$



Assuming cylindrical symm.,

$$\int_0^{2\pi} d\phi = 2\pi$$

Further, note $\int_0^{\pi} \sin\theta \, d\theta = \int_1^{-1} -d(\cos\theta)$

$$= \int_{-1}^{1} d(\cos\theta)$$

$$\therefore G(\vec{x}-\vec{x}') = \frac{2\pi}{(2\pi)^3} \int_0^{\infty} \frac{k^2 dk}{E_n - \frac{\hbar^2 |\vec{k}|^2}{2m} + i\epsilon} \times \int_{-1}^{1} d(\cos\theta) e^{ik|\vec{x}-\vec{x}'|\cos\theta}$$

Now $\int_{-1}^{1} du \, e^{i\beta u} = \frac{1}{i\beta}\left(e^{i\beta} - e^{-i\beta}\right) = \frac{2}{\beta}\sin\beta$

$$\therefore G(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2\,dk}{E_n - \frac{\hbar^2|\vec{k}|^2}{2m}+i\epsilon} \times \frac{2\sin(k|\vec{x}-\vec{x}'|)}{k|\vec{x}-\vec{x}'|}$$

$$= \frac{2}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{x}'|}\int_0^\infty \frac{k\,dk\,\sin(k|\vec{x}-\vec{x}'|)}{E_n - \frac{\hbar^2|\vec{k}|^2}{2m}+i\epsilon}$$

$$= \frac{2}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{x}'|}\frac{1}{2i}\cancel{\mathrm{Im}}\int_{-\infty}^\infty \frac{k\,dk\,e^{ik|\vec{x}-\vec{x}'|}}{\frac{\hbar^2k_n^2}{2m}-\frac{\hbar^2k^2}{2m}+i\epsilon}$$

Note denominator $-(k-\tilde{k})(k+\tilde{k}) = -(k-k_n-i\epsilon)(k+k_n+i\epsilon)$

Rough: $k^2-k_n^2-i\epsilon = \left(k-\sqrt{k_n^2+i\epsilon}\right)\left(k+\sqrt{k_n^2+i\epsilon}\right)$ $\sqrt{k_n^2+i\epsilon}=k_n\left(1+\frac{i\epsilon}{2k_n^2}\ldots\right)$

$\simeq (k-k_n-i\epsilon)(k+k_n+i\epsilon)$

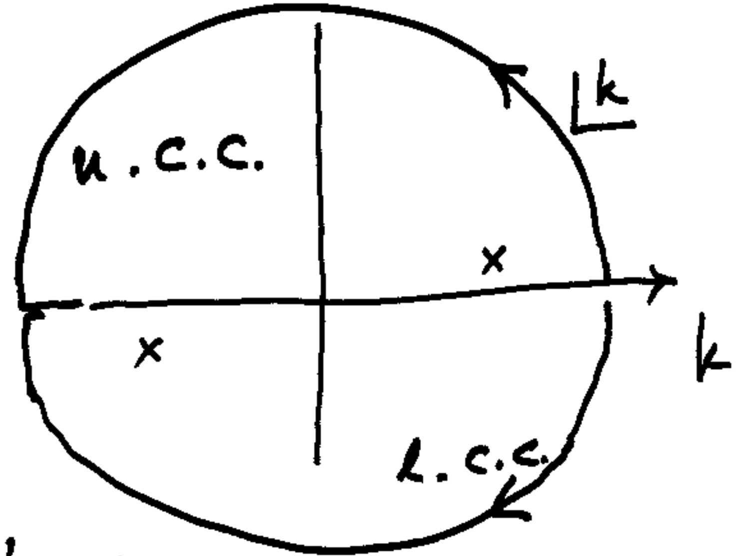
$\ldots\epsilon$ rescaled

u.s.c. $e^{i \times i|k||\vec{x}-\vec{x}'|}$

$\rightarrow e^{-|k||\vec{x}-\vec{x}'|}$

l.s.c. $e^{i(-i|k|)|\vec{x}-\vec{x}'|}$

$\rightarrow e^{|k||\vec{x}-\vec{x}'|}$



Thus we close the contour in the complex k plane ~~at~~ using the upper semi-circle (u.s.c.) and get the residue at the pole $k = k_n + i\epsilon$

$$\therefore G(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^2}\frac{1}{|\vec{x}-\vec{x}'|} \times \frac{2m}{i\hbar^2}\,\cancel{\mathrm{Im}}\; 2\pi\frac{i}{i}(-1)\frac{\cancel{k_n}\, e^{ik_n|\vec{x}-\vec{x}'|}}{2\cancel{k_n}}$$

$$= -\frac{1}{4\pi}\frac{e^{ik_n|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\frac{2m}{\hbar^2}$$

Standard statement:

$$(\nabla^2 + \alpha^2)\, G(\vec{x} - \vec{x}') = \delta^3(\vec{x} - \vec{x}')$$

has the solution

$$G(\vec{x} - \vec{x}') = -\frac{e^{i\alpha|\vec{x} - \vec{x}'|}}{4\pi\, |\vec{x} - \vec{x}'|}$$

Thus the solution of

$$(\nabla^2 + \alpha^2) f(\vec{x}) = F(\vec{x})$$

can be written as

$$f(\vec{x}) = f_{\text{homog.}}(\vec{x}) + \left(-\frac{1}{4\pi}\right) \int d^3\vec{x}' \, \frac{e^{i\alpha|\vec{x} - \vec{x}'|}\, F(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Recall Coulomb problem $\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = 4\pi \rho$

$$\therefore \ \phi = \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$