Scattering states - stationary Green Function (Sakurai)

$$\langle \vec{x} | \psi^+ \rangle = \langle \vec{x} | \phi \rangle - \frac{2m}{\hbar^2} \int d^3x' \, \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi |\vec{x}-\vec{x}'|} \cdot \langle \vec{x}' | V | \psi^+ \rangle$$

[Same as

$$\psi_f = \psi_i + \int G V \psi_i$$

specialised to in-out philosophy of S-matrix]

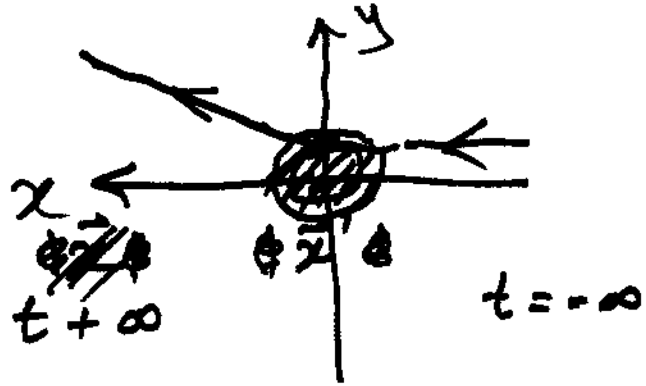
We next move towards :

① $\chi^+ = u + \frac{c}{r} f e^{ikr}$ ② $\frac{d\sigma}{d\Omega} = |f|^2$ ③ $f \sim \int V e^{i\vec{q}\cdot\vec{r}} d^3r$

$\vec{q} = \vec{k}_f - \vec{k}_i$

Simplification:

$|\vec{x}'| \ll |\vec{x}|$



In $\dfrac{1}{|\vec{x}-\vec{x}'|} \rightarrow \dfrac{1}{|\vec{x}|}$

In the exponent, note $|\vec{x}-\vec{x}'| = \sqrt{|\vec{x}|^2 + |\vec{x}'|^2 - 2\,\vec{x}\cdot\vec{x}'}$

$$\therefore |\vec{x}-\vec{x}'| = |\vec{x}|\left(1 - \frac{1}{\cancel{2}}\cancel{2}\frac{|\vec{x}'|}{|\vec{x}|}\cos\theta\right) = |\vec{x}| - \cancel{2}|\vec{x}'|\cos\theta$$
$$\equiv r - r'\cos\theta$$

Thus eqn. for scattered wave function

$$\langle\vec{x}|\psi^+\rangle = \langle x|\phi\rangle - \frac{2m}{\hbar^2}\int d^3x'\, \frac{e^{ikr}}{4\pi r}\, e^{-ikr'\cos\theta}\, V(\vec{x}')\langle\vec{x}'|\psi^+\rangle$$

Conventions: $\langle\vec{k}|\vec{k}'\rangle = \delta^3(\vec{k}-\vec{k}')$; $\langle\vec{x}|\vec{k}\rangle = \dfrac{e^{i\vec{x}\cdot\vec{k}}}{(2\pi)^{3/2}}$

$$\psi^+ \equiv \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}} + \frac{1}{(2\pi)^{3/2}} \frac{e^{ikr}}{r} f(\vec{k}_f, \vec{k}_i)$$

where f is identified as

$$\underset{\vec{k}'}{\overset{\equiv}{f(\vec{k}_f}}, \underset{\vec{k}}{\overset{\equiv}{\vec{k}_i}}) = -\frac{2m}{\hbar^2}\frac{1}{4\pi}(2\pi)^3 \underbrace{\int \frac{d^3x' \, e^{-i\vec{k}'\cdot\vec{r}'}}{(2\pi)^{3/2}} V(\vec{x}') \, \psi^+(\vec{x}')}_{\langle \vec{k}' | V | \vec{k} \rangle}$$

The lowest approx. to this iterative equation is $\psi^+(\vec{x}') = \phi(\vec{x}') = \frac{e^{i\vec{k}\cdot\vec{x}'}}{(2\pi)^{3/2}}$ $\longrightarrow$ Born approximation

Note $\int \frac{d^3x'}{(2\pi)^3} e^{-i\vec{k}'\cdot\vec{x}' + i\vec{k}\cdot\vec{x}'} V(\vec{x}) = \tilde{V}(\vec{k}-\vec{k}')$ or $\tilde{V}(\vec{q})$

Now define

differential cross-section $d\sigma = \dfrac{\left(\text{No. of events}/\text{time}\right)_{\text{in solid angle } d\Omega}}{\text{Incident flux}}$

$$= \frac{|\vec{j}_{scatt}| \times r^2 d\Omega}{|\vec{j}_{inc.}|}$$

$$= |f(\vec{k}', \vec{k})|^2 d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k}', \vec{k})|^2}$$

$$\boxed{f = -\frac{m}{\hbar^2} 4\pi^2 \tilde{V}(\vec{k}' - \vec{k})}$$ Born approx.

Example: Screened Coulomb potential
(Yukawa potential)

$$V(\vec{x}) = -\frac{Ze^2}{r} e^{-r/a}$$

$$\tilde{V}(\vec{q}) = \int \frac{e^{i\vec{q}\cdot\vec{r}}}{(2\pi)^3} d^3r \left(-Ze^2\right) \frac{1}{r} e^{-r/a}$$

$$= \frac{1}{(2\pi)^3}\left(-Ze^2\right) 2\pi \int_{-1}^{1}\int_{0}^{\infty} d(\cos\theta)\, e^{iqr\cos\theta}\, \frac{e^{-r/a}}{r} r^2 dr$$

$$= \frac{1}{(2\pi)^3}\left(-Ze^2\right) 2\pi \times \int \frac{1}{iqr} \cdot 2i \frac{\sin qr\, e^{-r/a}}{r} r^2 dr$$

$$= \frac{-Ze^2}{(2\pi)^2} \frac{2}{q} \int_0^{\infty} \sin qr\, e^{-r/a} dr =$$