S-matrix : Phase shifts

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IIT Bombay

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Recall, we proposed

$$\chi_\alpha^+ = \underbrace{u_\alpha(\vec{r})}_{\text{incoming wave}} + C\,\frac{e^{ik_\alpha r}}{r} f(\hat{r})$$

Form of the scattered wave with steady incoming flux (stationery situation)

In Born approx. $\underbrace{|\vec{r}\,'|}_{\text{Domain of } V(\vec{r})} \ll |\vec{r}| \longrightarrow$ Laboratory size

$$f(\vec{k}', \vec{k}) = -4\pi^2 \frac{m}{\hbar^2} \tilde{V}(\vec{q})$$

$\left\{ \begin{array}{l} \sim \text{ means fourier transf} \\ \vec{q} = \vec{k}' - \vec{k} \end{array} \right.$

$$\frac{d\sigma}{d\Omega} = |f(\vec{k}', \vec{k})|^2$$

"Phase shift" formalism:

Simplifying Assumptions about angular dependence $\cos\theta = (\vec{k}' \cdot \vec{k}) / |\vec{k}|^2$

note $|\vec{k}'| = |\vec{k}|$ elastic scattering

Consider incoming flux along the z axis as a plane wave:

$$u_\alpha(\vec{r}) = e^{ikz} \qquad \text{"}\alpha\text{"} \equiv |\vec{k}|$$

Using Legendre polynomials we have

$$e^{ikz} = e^{ikr\cos\theta} = \frac{1}{2ikr} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \left[(-1)^{l+1} e^{-ikr} + e^{ikr} \right]$$

We ~~compare~~ do this kind of expansion also for the "scattered" term in χ_α^+

Consider time indep. Schrödinger eqn. for the general form of ψ written as

$$\psi_k = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) R_{kl}(r)$$

$|\vec{k}|$ determines energy

Then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} R_{kl}\right) + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} V(r)\right] R_{kl} = 0$$

The solutions of this eqn. are known to be

$$R_{kl}^{\pm} = \pm i \sqrt{\frac{k\pi}{2r}}\, H^{(1)}_{l+\frac{1}{2}}{}^{(2)}(kr)$$

where $H^{(1)}, H^{(2)}$ are Hankel functions

Asymptotically, the $R_{k\ell}$ behave as $\frac{\cos kr}{r}$ & $\sin(kr)/r$ ... more specifically

$$R_{k\ell} \approx \frac{2}{r}\sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell\right)$$

where δ_ℓ the relative coeff. between sin & cos is to be determined (from boundary conditions)

$$\approx \frac{1}{ir}\left\{(-i)^\ell e^{i(kr+\delta_\ell)} - i^\ell e^{-i(kr+\delta_\ell)}\right\}$$

$\left\{ \begin{array}{l} \text{rewriting} \\ e^{\pm i\ell\pi/2} \end{array} \right.$

Thus,

$$\psi \approx \frac{1}{2ikr}\sum_{l=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\left\{(-1)^{\ell+1}e^{-ikr} + S_\ell e^{ikr}\right\}$$

$$S_\ell = e^{2i\delta_\ell}$$

i.e. we chose to write $A_\ell = \frac{1}{2k}(2\ell+1)\, i^\ell e^{i\delta_\ell}$

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Thus we get

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(S_l - 1) P_l(\cos\theta)$$

We shall later use $f_l(k)$ to be

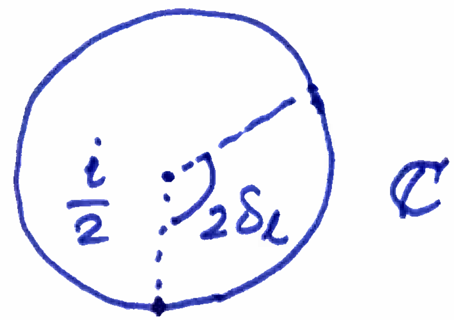
$$S_l = \boxed{e^{2i\delta_l} \equiv 1 + 2ikf_l(k)}$$

[A pictorial way of thinking

$$kf_l = \frac{1}{2i}\left(e^{2i\delta_l} - 1\right)$$

$$= \frac{i}{2} - \frac{i}{2} e^{2i\delta_l}$$

$$= \frac{i}{2} + \frac{1}{2}\left(e^{2i\delta_l - i\pi/2}\right)$$



Putting this $f(\theta)$ into our formula

$$d\sigma = 2\pi \sin\theta \, d\theta \, |f(\theta)|^2$$

$$\therefore \sigma = 2\pi \int_{-1}^{1} d(\cos\theta) \left| \sum_{\ell=0}^{\infty} (2\ell+1)(e^{2i\delta_\ell}-1) P_\ell(\cos\theta) \right|^2$$

we get a double sum $\sum_{\ell,\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \times \dots$

but we have $\int_0^\pi P_\ell^2 \sin\theta \, d\theta = \frac{2}{2\ell+1}$ and zero if $\ell \neq \ell'$

$$\therefore \boxed{\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell}$$

Can also be written $\sigma_\ell = 4\pi(2\ell+1)|f_\ell|^2$ & $\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell$