Phase shifts - calculation & example

(ref. Landau & Lifshitz vol 3 ; "QM Non-Rel Theory")

One general result :

"Im $f(\theta=0)$ reproduces Total cross section σ "

$$f(\theta=0) = \frac{e^{i\delta_l}}{k}\sum_{l=0}^{\infty}(2l+1)\left(\frac{e^{i\delta_l}-e^{-i\delta_l}}{2i}\right)\underbrace{P_l(1)}_{=1}$$

$$= \frac{e^{i\delta_l}}{k}\sum_{l=0}^{\infty}(2l+1)\sin\delta_l$$

$$\boxed{\text{Im} f(\theta=0) = \frac{k}{4\pi}\sigma_{tot}}$$

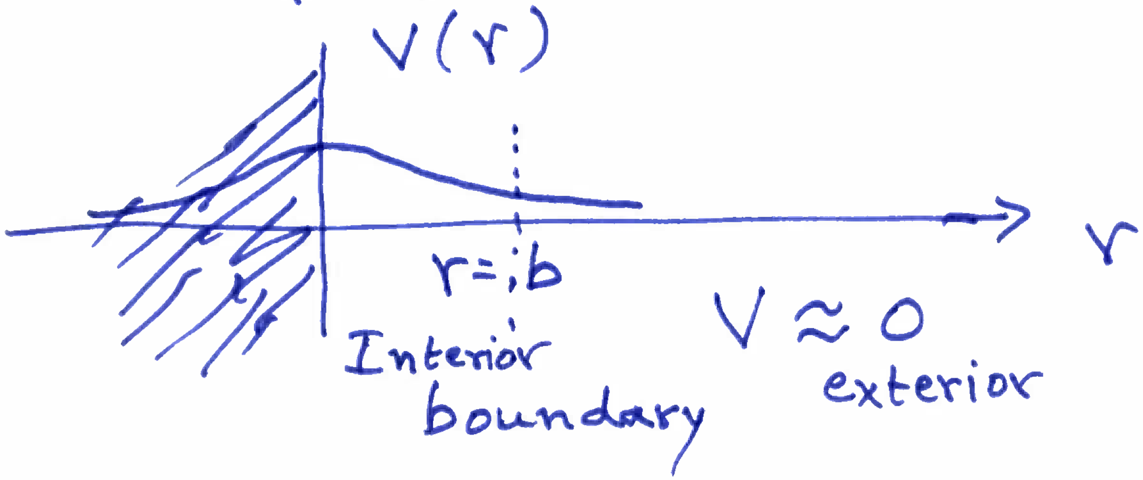
$\frac{\hbar^2 k^2}{2m}$ = ~~Kin.~~ energy. at $t = \pm\infty$

"Optical theorem" Consequence of unitarity

Calculation of phase shifts :

[Putting together L&L derivation with further development from Sakurai]

Philosophy



The problem of phase "shift" is solved by matching the interior & exterior wave functions at r = b

Integrating time indep. Schrö. eqn. from interior region, the logarithmic derivative $\psi'/\psi\,(r=b)$ carries the info. about $V(r)$ & is related to δ_ℓ

Consider writing (using $A_\ell R_{k\ell}$ introduced earlier)

[Sakurai : $A_\ell(r) \propto u_\ell(r)$; L&L "A_ℓ" & $R_{k\ell}$]

$$A_\ell R_{k\ell}(r) = c_\ell^{(1)} h_\ell^{(1)}(kr) + c_\ell^{(2)} h_\ell^{(2)}(kr)$$

$h_\ell^{(1)(2)}$ are $H^{(1)}_{\ell+\frac{1}{2}}$, $H^{(2)}_{\ell+\frac{1}{2}}$ but with diff. normalization

Define logarithmic derivative $\beta_\ell = \left[\frac{r}{A_\ell R_{k\ell}} \times \left(\frac{d}{dr} A_\ell R_{k\ell}\right)\right]_{r=b}$

Thus, rewrite $A_\ell R_{k\ell}$ in terms of δ_ℓ and then calculate β_ℓ

Math results : $\left\{ h_\ell^{(1)} \xrightarrow{r\to\infty} \frac{e^{i(kr-\frac{\pi}{2}\ell)}}{ikr} \qquad h_\ell^{(2)} \xrightarrow{r\to\infty} \frac{e^{-i(kr-\frac{\pi}{2}\ell)}}{ikr} \right.$

Make a transition to j_ℓ & n_ℓ :

$$h_\ell^{(1)} = j_\ell + i n_\ell \qquad h^{(2)} = j_\ell - i n_\ell$$

where j_ℓ are Bessel functions & n_ℓ Neumann functions

$$j_\ell(r) = \sqrt{\frac{2}{\pi r}}\, J_{\ell+\frac{1}{2}}(r) \; ; \qquad n_\ell = (-1)^{\ell+1}\left(\frac{\pi}{2r}\right)^{1/2} J_{-\ell-\frac{1}{2}}(r)$$

Relating to our δ_ℓ introduced earlier,

$$\boxed{A_\ell R_{k\ell}(r) = e^{i\delta_\ell}\left(\cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell\, n_\ell(kr)\right)}$$

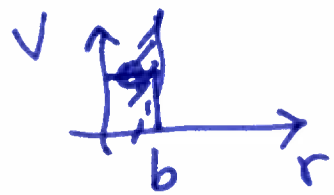
Thus $$\beta_\ell = kb\left(\frac{j_\ell'(kb)\cos\delta_\ell - n_\ell'(kb)\sin\delta_\ell}{j_\ell(kb)\cos\delta_\ell - n_\ell(kb)\sin\delta_\ell}\right)$$

This formula can be inverted to read

$$\tan\delta_\ell = \frac{kb\, j_\ell'(kb) - \beta_\ell\, j_\ell(kb)}{kb\, n_\ell'(kb) - \beta_\ell\, n_\ell(kb)}$$

Thus if β_ℓ can be calculated from the knowledge of the interior solution, δ_ℓ in the asymptotic region is determined.

<u>Example</u>: Hard sphere



$$V = \begin{cases} \infty & r < b \\ 0 & r > b \end{cases}$$

Need $A_\ell R_{k\ell}(b) = 0 \Rightarrow \cos\delta_\ell\, j_\ell(kb) - \sin\delta_\ell\, n_\ell(kb) = 0$

i.e. $\tan\delta_\ell = \dfrac{j_\ell(kb)}{n_\ell(kb)}$