Method of phase shifts – examples

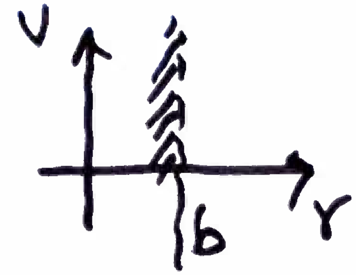
Clarification on convention:

indep. of r

We continue to use the form $\psi \sim A_\ell R_{k\ell}(r)$ as defined in slides 2, 3, 4 of lect 31 (as per L & L). $A_\ell(r)$ of Sakurai is same as our $A_\ell R_{k\ell}(r)$

Example of "hard sphere" (continued)

$V(r) = \infty$ for $r < b$; $V(r) = 0$ for $r > b$



$$\tan\delta_\ell = \frac{j_\ell(kb)}{n_\ell(kb)} \xrightarrow{kb \to 0} \frac{(kb)^{2\ell+1}}{\{(2\ell+1)((2\ell-1)!!)^2\}}$$

$$l = 0: \quad \tan\delta_0 = \frac{\sin kb/b}{-\cos kb/b} = -\tan kb$$

$$\text{Now} \quad \frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} \xrightarrow{kb\to 0} \frac{k^2 b^2}{k^2} = b^2$$

$$\text{recall} \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \boxed{\sum_{l=0}^{\infty} 4\pi(2l+1)|f_l|^2 = \frac{d\sigma}{d\Omega}}$$

$$\text{where} \quad e^{2i\delta_l} = S_l \equiv 1 + 2ikf_l(k)$$

$$\text{i.e.} \quad f_l(k) = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \boxed{e^{i\delta_l}\frac{\sin\delta_l}{k} = f_l(k)}$$

$$\therefore \sigma_{tot} = 4\pi b^2$$

→ This is expected of diffraction of plane wave from small sphere but disagrees with geometrical optics $\sim \pi b^2$

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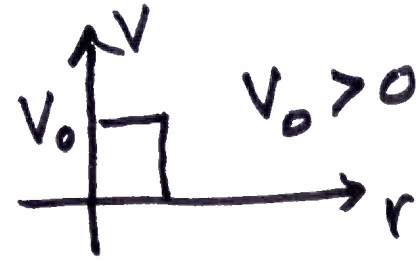
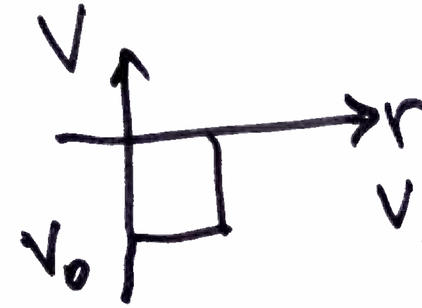
Example 2: Finite size barrier/well

$$V(r) = \begin{cases} V_0 & r < b \\ 0 & r > b \end{cases}$$



$V_0 > 0$



$V_0 < 0$

For $r > b$ we have the solution of the form as before

$$A_\ell R_{k\ell}(r) = e^{i\delta_\ell}\left(j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell\right)$$

$$\simeq e^{i\delta_\ell}\,\frac{\sin(kr + \delta_\ell - \pi\ell/2)}{kr}$$

$$j_\ell \sim \frac{\sin}{r}, \quad n_\ell \sim \frac{\cos}{r}$$

[Lect 31 pg 4]

The δ_ℓ is then found from logarithmic derivative of the interior wave function

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Interior wave function:

Let $u_\ell(r) = r \times A_\ell R_{k\ell}(r)$ → redefine from Lect 31 pg 3

Then $u_\ell'' + \left\{ k^2 - \frac{2m}{\hbar^2} V(r) - \frac{\ell(\ell+1)}{r^2} \right\} u_\ell = 0$

$$\left[\text{Check} : \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \frac{u(r)}{r} \right) = \frac{1}{r} \frac{d^2}{dr^2} u \right.$$

Using $\ell = 0$ and $V(r) = V_0$ // Let $k_1^2 = k^2 - \frac{2m}{\hbar^2} V_0$

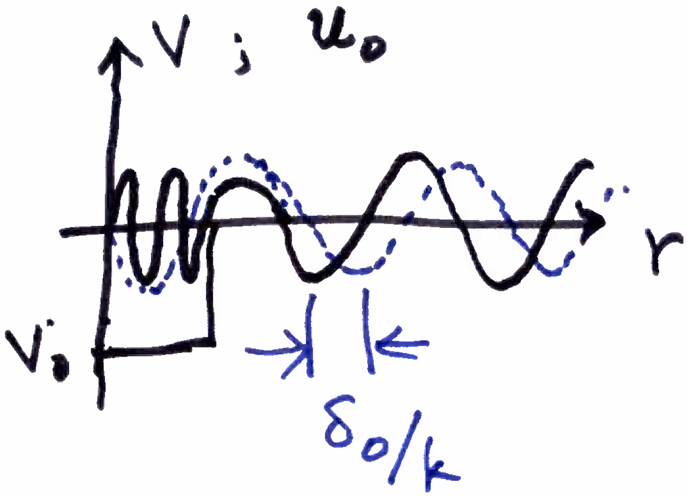
$u_0 = r A_0 R_{k0}(r) = c_1 \sin k_1 r + \cancel{c_2 \cos k_1 r} \to 0$ $u = 0$ at $r = 0$

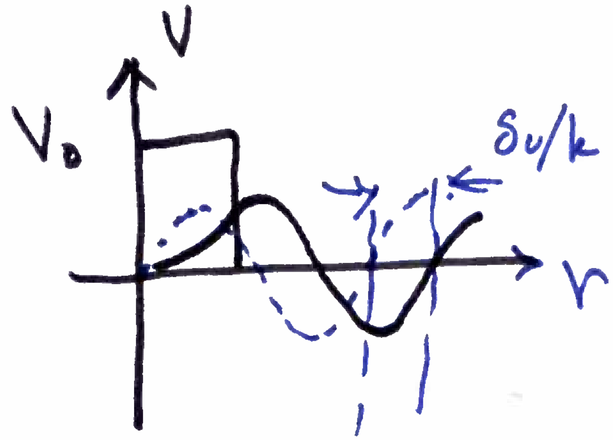
Thus

interior	exterior
$\frac{1}{kr} \sin k_1 r$ // $\frac{\sinh(k_1 r)}{kr}$	$\frac{1}{kr} \sin(kr + \delta_0)$

k_1 real // k_1 imaginary

Qualitative picture





Since $\sigma \propto \sum \sin^2\delta_l \xrightarrow{kb\to 0} \sin^2\delta_0$

max scatt. $\delta_0 = \frac{\pi}{2}$

"no scatt" $\delta_0 = \pi$