

# Proof of Path integral formula

Denote basis states as  $|q\rangle$

... equivalent other basis  $|p\rangle$

$$q_{\text{op}} |q\rangle = q |q\rangle ; \quad \langle q' | q \rangle = \delta(q - q')$$

$$p_{\text{op}} |p\rangle = p |p\rangle ; \quad \langle p' | p \rangle = \delta(p - p')$$

Thus wavefunction  $\psi(q, t) = \langle q | \psi(t) \rangle$



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Q. Kinematics:

$$\langle q | p \rangle = \frac{e^{iqp/\hbar}}{\sqrt{2\pi\hbar}}$$

This is equiv. to  $p_{op} = -i\hbar \frac{d}{dq}$  Schrö.

or  $[q_{op}, p_{op}] = i\hbar$  Heis.

Propose instantaneous basis states

$$q_H(t) |q,t\rangle_D = q |q,t\rangle_D$$

↳ "Dirac picture"



... why  $|q_t\rangle_D$  ?

$$|q_t\rangle_s = e^{-iHt/\hbar} |q_0\rangle$$

$$= e^{-iHt/\hbar} |q\rangle_H$$



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Thus the relation of Schr. & Heis. pictures

$$\mathcal{O}_H(t) = e^{iHt/\hbar} \mathcal{O}_s e^{-iHt/\hbar}$$

$$\begin{aligned} \text{Now note, } q_H(t) |q_t\rangle_D &= e^{iHt/\hbar} q_s e^{-iHt/\hbar} |q_t\rangle_D \\ &= e^{iHt/\hbar} q_s \underbrace{e^{-iHt/\hbar} e^{iHt/\hbar}}_{\text{1}} |q\rangle \\ &= q e^{iHt/\hbar} |q\rangle = \cancel{\text{1}} \cancel{|q\rangle} q |q_t\rangle_D \end{aligned}$$

Now we recast the statement  
of time evolution using  $|q,t\rangle_D$ :

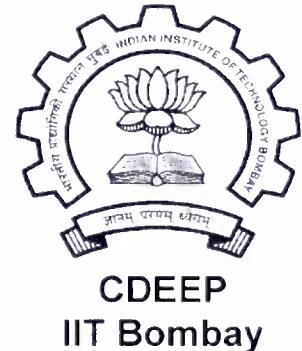
$$\psi(q_f, t_f) = \langle q_f | \psi(t_f) \rangle_S = \langle q_f | \psi \rangle_H$$

$$\downarrow \langle q_f | e^{-iHt_f/\hbar} | \psi_0 \rangle$$

Now we can use this to write

$$\psi(q_f, t_f) = \int dq_i \langle q_f | q_i, t_i \rangle_D \langle q_i | \psi \rangle_H$$

$$\text{using } \Pi = \int dq_i |q_i, t_i\rangle_D \langle q_i | \quad t_f > t_i$$



$$\psi(q_f, t_f) = \int dq_i K(q_f, t_f; q_i, t_i) \psi(q_i, t_i)$$

with  $K(q_f, t_f; q_i, t_i) \equiv \langle q_f, t_f | q_i, t_i \rangle_D$

$t_f > t_i$



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called the kernel or Green's Function

also can be called "propagator".

Now

$$\langle q_f, t_f | q_i, t_i \rangle_D = \langle q_f | e^{-i H(t_f - t_i)/\hbar} | q_i \rangle$$

Can we calculate  $\langle q, t + \Delta t | q, t \rangle_D$  for small  $\Delta t$ ?

... can be compounded into final answer

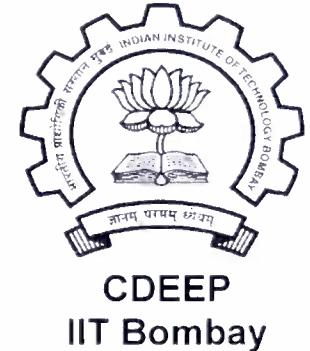
... a useful different object

$$\langle p|q\rangle_D = \langle p|e^{-iH\Delta t/\hbar}|q\rangle$$
$$\approx \langle p|1 - i\frac{H\Delta t}{\hbar} + O(\Delta t^2)\dots|q\rangle$$

Normal ordering in  $H$ :

$H(q, p)$  written with all  $p$  on the left of all  $q$ 's

Thus  $H$  above can be replaced by its value



$$\langle p_{t+\Delta t} | q_t \rangle \approx \frac{e^{-iq_p \Delta t / \hbar}}{\sqrt{2\pi\hbar}} - \frac{i}{\hbar} H(p, q) \Delta t \frac{e^{-iq_p \Delta t / \hbar}}{\sqrt{2\pi\hbar}}$$

$$\sim \frac{e^{-iq_p \Delta t / \hbar}}{\sqrt{2\pi\hbar}} \left\{ 1 - \frac{i}{\hbar} H(p, q) \Delta t \dots \right\}$$

$$\sim \frac{e^{-iq_p \Delta t / \hbar}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} H(q, p) \Delta t} \dots \text{if } \Delta t \text{ is small}$$



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Thus, consider

$$\langle q_f t_f | q_i t_i \rangle_D = \underbrace{\int dP_f}_{\frac{e^{i(q_f p_f - q_i p_i)/\hbar}}{\sqrt{2\pi\hbar}}} \underbrace{\langle q_f t_f | p_f t_f \rangle_D \langle p_f t_f | q_i t_i \rangle_D}_{\text{from previous pg. if } t_f - t_i \text{ is small}}$$

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suggests  $\rightarrow$

$$\sim \int \frac{dp}{2\pi\hbar} e^{i(q_f p_f - q_i p_i)/\hbar - iH\Delta t/\hbar}$$

$$\sim \int \frac{dp}{2\pi\hbar} e^{i p_f \frac{\Delta q}{\Delta t} \Delta t/\hbar - iH \frac{\Delta t}{\hbar}}$$

$$\sim \int dp e^{i(p_f - H) \frac{\Delta t}{\hbar}}$$

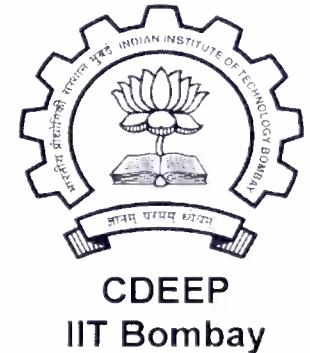


Introduce

$$q_1, q_2 \dots q_N ; p_1, p_2 \dots p_N, p_{N+1} = p_f$$

at times  $t_i < t_1 < t_2 \dots < t_N < t_f$

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$$\begin{aligned} D \langle q_f t_f | q_i t_i \rangle_D &= \int \langle q_f t_f | p_{N+1} t_f \rangle \langle p_{N+1} t_f | q_N t_N \rangle \times \\ &\quad \langle q_N t_N | p_N t_N \rangle \langle p_N t_N | q_{N-1} t_{N-1} \rangle \times \\ &\quad \dots \dots \times \\ &\quad \times \langle q_i t_i | p_i t_i \rangle \langle p_i t_i | q_i t_i \rangle \\ &\quad \times \prod_1^N dq_i \prod_1^{N+1} dp_i \end{aligned}$$

As the number of slices  $N \rightarrow \infty$ ,

we get  $\Delta t_N = t_N - t_{N-1} \rightarrow 0$

and our formula becomes



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$$\lim_{N \rightarrow \infty} \int \frac{dP_{N+1}}{\sqrt{2\pi k}} \prod_{j=1}^N \frac{dq_j dp_j}{2\pi k} \exp \left\{ i \sum_j \left[ P_{j+1} (q_{j+1} - q_j) - H(P_j, q_j) \Delta t_j \right] \right\}$$

Symbolically we write

$$\boxed{\underbrace{\int Dp Dq \exp \left\{ i \int_{t_i}^{t_f} dt (p \dot{q} - H(p, q)) \right\}}$$

integration over paths