Path integral formula (continued)

Remark:

- Path Integral goes far beyond multivariate calculus integration
- Some formal devepments due to Stratanovich & C. deWitt
- Analogy to heat diffusion equation

We need to use a suggestive formula:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/\sigma^2} = \sqrt{\pi}\,\sigma \qquad \Bigg| \qquad \int_{-\infty}^{\infty} dx\, e^{i x^2/\sigma^2} = \sqrt{i\pi}\,\sigma$$

This formal prescription correctly gives parametric dependence of the answer on the constants in the integral.

Lagrangian version:

Note that usually $H = \frac{p^2}{2m} + V(q)$

more generally, $\sum_{ab} A_{ab}(q_c)\, p_a p_b + \sum_a B_a(q_c)\, p_a$

... some quadratic form & even linear terms with this form of H, a typical integral is

$$\int \frac{dp_j}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar}\left(p_j \Delta q_j - \frac{p_j^2}{2m}\Delta t_j\right)}$$

Complete the square :

$$\underbrace{\frac{p^2}{2m}\Delta t_j - p_j \Delta q_j + \frac{\Delta q_j^2 m}{2\Delta t_j}}_{\frac{\Delta t_j}{2m}\Big(\underbrace{p_j - \frac{\Delta q_j m}{\Delta t}}_{\tilde{p}_j}\Big)^2} - \frac{\Delta q_j^2 m}{2\Delta t_j}$$

$$\frac{\Delta t_j}{2m}\Big(\underbrace{p_j - \frac{\Delta q_j m}{\Delta t}}_{\tilde{p}_j}\Big)^2 - \frac{\Delta q_j^2 m}{2\Delta t_j}$$

Thus integral becomes

$$\int \frac{d\tilde{p}_j}{\sqrt{2\pi\hbar}}\, e^{-\frac{i}{\hbar}\frac{\Delta t_j}{2m}\tilde{p}_j^2} \cdot e^{\frac{i}{\hbar}\frac{m}{2\Delta t_j}\Delta q_j^2} = e^{\frac{i}{\hbar}\frac{m}{2\Delta t_j}\Delta q_j^2} \times \sqrt{\frac{2m\pi\hbar}{i\Delta t_j}} \times \frac{1}{\sqrt{2\pi\hbar}}$$

Thus,

$$\int \frac{dp_{N+1}}{\sqrt{2\pi\hbar}} \int \prod_{1}^{N} \left[\frac{dp_j\, dq_j}{2\pi\hbar}\right] \exp\left\{\frac{i}{\hbar}\sum_j p_j\,(q_{j+1}-q_j) - \frac{p_j^2}{2m}\Delta t_j - V(q_j)\Delta t_j\right\}$$

$$= \int \prod_{1}^{N} \frac{dq_j}{\sqrt{2\pi\hbar}} \sqrt{\frac{m}{i\Delta t_j}}\; e^{\left\{\frac{i}{\hbar}\sum_j \frac{m}{2\Delta t_j}\Delta q_j^2 - \frac{i}{\hbar}\sum_j V(q_j)\Delta t_j\right\}}$$

Suggestively, $\mathcal{N}\int \mathcal{D}q\; e^{\frac{i}{\hbar}\int_{t_i}^{t_f}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)dt}$

Compatibility with Schrödinger Eqn.

Note the we are looking for the kernel K in the formula

$$\psi(x_f t_f) = \int K(x_f t_f; x_i t_i)\, \psi(x_i t_i)\, dx_i$$

Now consider

$$\psi(x_f t_f) = \sum_n c_n e^{-iE_n(t_f - t_i)/\hbar} \varphi_n(x_f)$$

where E_n are the eigenvalues of H and φ_n are the corresponding eigenfunctions, normalised

Now, $\psi(x_i t_i) = \sum_n c_n \varphi_n(x_i)$ .... setting $t = t_i$

$$\therefore\ C_n = \int dx_i\, \psi(x_i t_i)\, \varphi_n^*(x_i)$$

$$\therefore\ \psi(x_f t_f) = \sum_n \left(\int dx_i\, \psi(x_i t_i)\, \varphi_n^*(x_i) \right) e^{-iE_n(t_f - t_i)/\hbar}\, \varphi_n(x_f)$$

$$= \int dx_i \left[\sum_n e^{-iE_n(t_f - t_i)/\hbar}\, \varphi_n(x_f)\, \varphi_n^*(x_i) \right] \psi(x_i t_i)$$

Thus we identify

$$K(x_f t_f ; x_i t_i) = \sum_n e^{-iE_n(t_f - t_i)/\hbar}\, \varphi_n(x_f)\, \varphi_n^*(x_i)$$

Trick for doing $\int dq_j \, e^{i \Delta q^2 \cdots}$

Let $y_j = q_{j+1} - q_j \qquad j = 1 \ldots N$

$$\sum y_j = q_f - q_i$$

$$\int \prod dq_j \longrightarrow \int \prod dy_j \, \underbrace{\delta\left(q_f - q_i - \sum y_j\right)}_{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, e^{ik\left(q_f - q_i - \sum_j y_j\right)}}$$