Mathematical secrets of the Universe

42 and therebefore :

“The answer was 42” thus starts the classic Hitchhikers Guide to the Galaxy. The fourth generation computer to be invented in some future century finally produced the answer to a problem posed centuries earlier; by which time the question has been forgotten. The author is intentionally cruel in choosing this most unremarkable number, in order that the parody would not be missed on us. It is a number with three prime factors, an even number to boot.

Theoretical physicists also believe that the secrets of the Universe will be delivered to them on a platter; except that they expect the answer to be both awefully clever and terribly simple, both at the same time. And ever since General Theory of Relativity, Physicists are big believers in shapes rather than numbers, Geometry rather than Analysis, and as such they seek this holy grail of a secret in some horribly complicated shapes in six dimensions (Kalabi-Yau manifolds of complex dimension three), albeit as solutions of a terribly simple differential equation, at least in its structure if not in its detail.

Why should one expect Mathematics to be so relevant to Physics? Let us recall the great “Eureka” experiment. Archimedes was the first to define a concept which is obvious in hindsight but is technical in its definition. Who would think of dividing weight by volume, two dimensionally unrelated quantities? But it was through this that he came to a clear criterion for floatation. This is a first clear use of numbers to arrive at a physical law. Archimedes is also credited with the laws of the lever, and with his audacious statement about lifting the earth by the arm of a lever. Here again we multiply a force and a distance. Much later it would become angular momentum. The most audacious attempt of Archimedes though was written in a memoire, addressed to his patron, in which he estimated how many grains of rice (or something similar) it would take to fill the Universe [1]. Fortunately Archimedes lived in the happy times when the Universe was much smaller than our estimates of its now. His Universe extends a few hundred miles above the earth, surprisingly comparable to the height of the atmosphere. (And the interior of the earth is gratuitously overlooked). Much later in our time, Fermi was to become famous for making audacious estimates of all kinds of numbers, through childish experiments and guesswork method which have become known by his name.

Mechanics and Continuum Mechanics

The Greeks also had formulae for perimeters and volumes of various interesting shapes. Thus quantification of geometry had certainly begun a long time ago. But the philosophy that set the stage for the modern applicability of mathematics to physics may be traced to Descartes in 1550’s when he unified numbers and geometry, providing analytical basis to geometry. The developers of calculus, notably Newton in 1660’s essentially built upon this approach to convert dynamics into rates of change. Physical system was geometrically ambedded in space and time, the latter were expressible as cartesian continuua, labelled by his method, and so dynamics became analysis.

One may observe at this point that the mathematics of Infinitesimal Calculus was indeed motivated by Physics, so it should not be a surprise that it was iminently applicable to Physics. Well, the plot thickens and the story gets longer, so that only when one finally arrives in the twentieth century that one is surprised to find branches of Physics suddenly becoming explicable by Mathematics developed in completely different contexts, as we will see below.

Newton proved using his Calculus that the $1/r^2$ law leads to a differential equation whose solution correctly reproduces Kepler’s laws. A large body of mathematics subsequently developed to solve the problems of point particle mechanics as also of fluid mechanics, elasticity, vibrations and sound and light propagation. A large systematic body of development here was the development of partila differential equations and of special functions, clearly acknowledged as inspired by Mathematical Physics. Among other landmark developments in Mathematics of the nineteenth century, relevant to our topic was Differential Geometry, which at the time of its development had little or no application to Physics. And these are the topics worth thinking of as we recount the relevance of Mathematics to Physics.

General Theory of Relativity

Come twentieth century and the first sign of Mathematics serving as a key to Physics was Einstein’s application of Differential Geometry to Gravity. Einstein had an uncanny insight by which he could reduce complicated situation to a few simple formulae relating the essential quantities. His disarmingly simple relationship between change in rest mass and available energy is all too well known. But faced with the challenge of mathematically implementing the equivalence of inertial mass and rest mass of particles, he needed to resort to more sophisticated Mathematics. Fortunately, Riemann, generalising Gauss had developed just the concepts in 1870's, needed to accommodate the new Physics Einstein was thinking of. More importantly, while Riemann had left behind a philosophical bluprint, Lévi-Civita just in the years 1902 to 1908 had developed the full fledged apparatus of covariant differential calculus. Einstein had in Marcel Grossmann a "mathematical assistant" who could understand these methods and assisted Einstein in formulating his hypotheses. The two of them worked closely until 1914 after which Grossmann left the collaboration. It was in 1915 that Einstein finally succeeded in arriving at the equations of General Relativity, entirely from a requirement of internal consistency within the paradigm of Differential Geometry.

It is interesting that David Hilbert the famous mathematician had one assistant assigned to each of the major branches of mathematical physics, Mechanics, Electromagnetism, Fluid Mechanics, etc, because he thought "Physics was too important to be left to Physicists". Hilbert through his assistants was aware of Einstein's valiant attempts at guessing the laws of General Relativity purely from mathematical reasoning. In 1908 Einstein made the first prediction of General Relativity, the bending of light by a heavy gravitating object. But this only gave the influence of gravity on test particles, independent of their rest mass. What determined Gravity's own dynamics was not yet known. Einstein and Grossmann faced deep conceptual difficulties which were communicated to the Prussian Academy in a series of three-monthly reports. Hilbert was thus aware of the technical problems Einstein faced. In 1915 Hilbert therefore arrived at the equations of General Relativity five days before Einstein [2], by a different route, indeed a route that should have been more familiar to a Physicist. Hilber could find the complex set of differential equations be varying an action principle, which could contain only a scalar. While Einstein struggled with the full tensor structure of the differential equations, there was a unique scalar in Riamannian geometry which Hilbert could exploit to arrive at the same equations.

Quantum Mechanics

But more was yet to come. Quantum Mechanics seemed to demand considerably greater sophistication than was used by Physicists until then. Matrix algebra was not a part of a physicist's standard training at that time. Heisenberg had to equip himself with these methods to formulate his theory. This was a purer and more direct insight into the fundamental Principle of Linear Superposition which underlies all of Quantum Mechanics and forms its backbone. For historical reasons however, the matrix methods could not be so easily received. A year later, Schrödinger came out with Wave Mechanics, which became much more popular, even causing some vexation to Heisenberg. Schrödinger implemented Superposition Principle in conjunction with de Broglie's hypothesis of waves associated with massive particles. de Broglie actually believed in pilot waves, present over and above the particle itself in the classical sense, and accompanying the latter like pilot motorcycles in front of dignitaries' motorcades. No such waves have ever been found and this great historic hint has only left behind a trail of confusion under the title "wave particle duality". As dirac clarified in his textbook treatment, there was no such duality. In retrospect we can say that de Brogli's observation amounted to associating a length scale, a "wavelength", $\lambda = h/p$ to momentum value p because Planck's constant h provided a natural dimensional conversion. Schrödinger could propose a *linear* equation giving space-time evolution of *superposition* of momentum eigenstates due to the underlying Superposition Principle. In the wave picture however this key principle of Quantum Mechanics somehow gets obfuscated in popular understanding by the hopelessness induced by the visual effect of a "spreading wave packet" in contrast to a classical point particle trundling along a Newtonian trajectory. This grand historical misunderstanding feeding the mills of philosophical writing, indeed challenging the completeness of Quantum Mechanical understanding of nature, deserves a separate treatment.

Wave Mechanics suddenly brought to life again all the special functions of Mathematical Physics. Originally developed to tackle completely different problems, now they were summoned to determine eignevalues of a variety of quantum Hamiltonians. More interestingly, the symmetries of the dynamics had to now be implemented through their realisation as *linear* representations of various classically known groups. Group theory was developed in Mathematics by Galois and Abel to tackle problems internal to Mathemtics, the solution of polynomial equations etc. Rotation group was a focus of Felix Klein's programme of classifying algebraic invariants as a way of understanding geometry. All of a sudden these methods arcane to Physics were in the mainstream. The proponent of these methods, Eugene Wigner was himself led to remark that there is a "rather unreasonable applicability of Mathematics to Physics."

Fibre bundles

A grand synthesis of all these ideas was arrived at in what are now called Yang-Mills theories. This framework has been successful in providing a description of three of the four fundamental interactions of nature. Only Gravity is different, but it also obeys the same paradigm as we describe now. Yang-Mills theory, also known as Fibre bundles, is a uification of Group Theory with Differential Geometry. The basic tenet of Differential Geometry is that a general curved space is nevertheless flat in a sufficintly small neighbourhood of a given point. This is called the local tangent space. Further, it requires that there should be a preferred way of transporting vectors from one tangent space to another. The latter idea can be generalised, to apply not only to vectors, but also to linear spaces that are representation spaces of a group.

In 1935 Heisenberg proposed a very far reaching idea, of isotopic spin. Thus protona nd neutron are some abstract projections of the same basic particle species called nucleon. In mathematical language this doublet of particles furnishes a representation of the group $SU(2)$ (isomorphic to the group of spins). It is to representations of this kind that the above mentioned principle of fibre bundles is applied. In modern understanding, electron and neutrino also form such a doublet representation and quarks of three "colors" constitute the triplet representation of $SU(3)$. Such representations are not in the obvious tangent space of a curved space, but are still treated as being on par, and are called "internal" parts of the tangent space.

The analogy to General Relativity is as follows. In General Relativity Einstein could show that the dynamics of Gravity is determined if we demand that it is invariant under the most general change of coordinate systems. In Special Relativity the freedom in the choice of frames of reference is exercised once, and is assumed to remain unchanged throughout all space-time. This was generalised to freedom of independent choice of frames from point to point in space-time, and was called the Principle of General Covariance. The Yang-Mills principle giving rise to fibre bundles can be stated as the freedom in the choice of the basis of the representation spaces (say for isospin multiplets) from point to point. This requirement alone fixes all the interactions of the concerned particles and their force fields. General Relativity and Yang-Mills have a lot in common conceptually but differ in essential mathematics, in that the freedom of frames in General Relativity concerns space-time frames, while for Yang-Mills it concerns "internal" frames.

If Hilbert was keeping tabs on Physics around the turn of 1900, then Élie Cartan the influential French mathematician was keeping tabs on it in the first half of the twentieth century. We do not have the space to even start mentioning Cartan's influence on modern mathematical physics. Suffice it to point out that Cartan had delivered lectures on the spinor representations of the Lorentz group in 1915, before spin was discovered in 1924, and about 15 years before Dirac found them by accident while obtaining relativistic wave equation of the electron. Cartan also went on to clarify some key conceptual issues with General Covariance as used by Einstein (and as stated above). One of Cartan's students, Ehresmann in 1956 proposed the notion of manifolds (curved spaces) whose local tangent spaces could also be Lie groups, and dubbed the framework "fibre bundles". Remarkably, this was almost simultaneous and independent of the proposal of the same concept by Physicists, Yang and Mills. The discovery of this far reaching notion was thus simultaneous in Mathematics and Physics.

Conclusion

In the second half of the twentieth century, after the success of Yang-Mills theories, theoretical research has been driven by a longing to see every law of Physics emerging from one unifying principle and adhering to one unitary framework. The early hints to this effect coming from Superstring Theory have not been borne out, at least not in detailed description of physical phenomena. However they have provided interesting mathematical laboratory for verifying various possibilities within Quantum Mechanics that would be difficult to properly compute in realistic theories.

We may also think that mathematicians are thus left far behind, in the paradise of a continuum world, which is not entirely free of its own pitfalls. Physicists on the other hand have moved on, recognised the discreteness that sets in at the microscopic scale, and the Superposition Principle that takes over in that domain. All the Quantum Mathematics, so to speak, has been developed entirely by physicists. It is in many of its workings, quite imperfect and impressionistic, and not rigorously understood. But as history has borne out, the rigour will be brought in at a due stage.

How long will this saga continue? Will all theories of physics become amenable to a geometric picture duly generalised? Or all geometry being captured in intricate mathematical functions like Jacobi Theta-function and Riemann Zeta-function will reduce to a menagerie of theorems in Number Theory? In the 44th chapter of their famous book [2] MTW speculate on the final nature of physical laws and conjecture that it will all ultimately reduce to very limpid logic of zeros and ones, may be a supercompact 1000th generation (that's eighth in binary) computing machine churning out fates of collective quantum systems. What they did not anticipate was that computing itself will move to quantum devices. While it may all reduce to efficient machine computation at some level, nothing can replace the joy of having grasped the essentials as a simple mathematical principle, and that is the partnership of Physics and Mathematics.

U. A. Yajnik
Physics Department
IIT Bombay

References

1. "God created the integers : the mathematical breakthroughs that changed history"[1] by S. W. Hawking (2007)

2. In history boxes in the textbook "Gravitation" by C. W. Misner, K. S. Thorne, and J. A. Wheeler (1973)

[1] "God created the integers, all the rest is the work of Man." - Leopold Kronecker, mathematician.